Uniformly distributed load

$$\psi = \phi \doteq [(1 - 0.014\alpha)/(1 - \alpha)] \qquad (15)$$

Uniformly varying load

$$\psi \doteq [(1 - 0.076\alpha)/(1 - \alpha)] \qquad \text{load} = w$$

$$\phi \doteq [(1 + 0.056\alpha)/(1 - \alpha)] \qquad \text{load} = 0$$

$$(16)$$

The accuracy of these equations is significantly better than Eqs. (9) and (10) because of the integrations involved.

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Stress-Intensity Factors for Longitudinal Shear Cracks

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THE concept of crack-tip stress-intensity factors applied to predictions of the fracture strength of cracked cylinders having finite cross sections has been discussed in previous work.[1-3] In this note, the cross sections are assumed to be infinite in extent owing to loads directed along the generators of the cylinder. This problem is of special interest, since it enables the consideration of exact solutions of many configurations, which are inaccessible for cracks under plane extension. Moreover, for cracks under longitudinal shear, it is possible to obtain results that will reveal certain qualitative effects common to all modes of crack surface displacements.

If an infinite body is subjected to longitudinal shear loads, the nonvanishing stress components may be expressed in complex form

$$\tau_{rt} - i\tau_{\theta t} = G(z/\bar{z})^{1/2} f'(z) \qquad (1)$$

where r and θ are polar coordinates, t is the coordinate axis directed along the center line of the cylinder, and G is the shear modulus of elasticity. In view of Eq. (1), the state of stress depends only upon the knowledge of a single function $f(z)$ of the complex variable $z = x + iy$.

In order to apply the current fracture mechanics theories to bodies containing crack-like fault lines, it is necessary to obtain the stress distribution near a crack point. For this purpose, an auxiliary complex plane ζ is introduced so that

$$z = \omega(\zeta) \qquad (2)$$

maps the crack configuration in the z plane onto the unit circle $|\zeta| = 1$ in the ζ plane. With the aid of Eq. (2), the right-hand side of Eq. (1) is transformed into

$$\tau_{rt} - i\tau_{\theta t} = G \left[\frac{\omega(\zeta)}{\overline{\omega(\bar{\zeta})}} \right]^{1/2} \frac{F'(\zeta)}{\omega'(\zeta)} \qquad (3)$$

where

$$f'(z) = \frac{df}{d\zeta}\frac{d\zeta}{dz} = \frac{F'(\zeta)}{\omega'(\zeta)}$$

To simplify the analysis, consider the mapping function $\omega(\zeta)$ for a single crack of length $2a$ centered at the origin. It is given by

$$\omega(\zeta) = \frac{a}{2}\left(\zeta + \frac{1}{\zeta}\right) \qquad \zeta = \frac{z + (z^2 - a^2)^{1/2}}{a} \qquad (4)$$

The crack tips $z = \pm a$ correspond to $\zeta = \pm 1$ on $|\zeta| = 1$. By way of a translation of the coordinate axes, $z - a = re^{i\theta}$, and restricting attention to a small region around the crack point, where r is small compared to a, Eq. (3) may be written as

$$\tau_{rt} - i\tau_{\theta t} = -[ik_3/(2r)^{1/2}]e^{i\theta/2} + 0(r^{1/2}) \qquad (5)$$

in which k_3 is a real parameter. In the course of deriving Eq. (5), $F'(\zeta)$ is assumed to be holomorphic on $|\zeta| = 1$, i.e., the limit of $F'(\zeta)$, as ζ approaches unity, is $F'(1) = -ik_3a^{1/2}/G$. Here, $F'(1)$ is chosen to be purely imaginary so as to satisfy the free surface conditions $\tau_{\theta t} = 0$ for $\theta = \pm\pi$. Now, separating the real and imaginary parts of Eq. (5), the results are

$$\tau_{rt} = [k_3/(2r)^{1/2}] \sin(\theta/2) + 0(r^{1/2})$$

$$\tau_{\theta t} = [k_3/(2r)^{1/2}] \cos(\theta/2) + 0(r^{1/2})$$

$$(6)$$

These equations were obtained earlier by Irwin,[4] who used a different approach. It is of fundamental interest to note that the inverse square-root characteristic of the stress singularity is actually imbedded in the mapping function and is independent of all the other conditions in the problem. Hence, the factor $1/r^{1/2}$ must, of necessity, be common to all crack problems where mapping of the type shown in Eq. (4) is employed. In fact, the conformal mapping technique may be applied to solve a variety of crack problems such as in the cases of plane extension and plate bending.

Now, equating Eqs. (1) and (5) and remembering that Eq. (5) holds only in the limit as z approaches z_1, the crack tip, the stress-intensity factor k_3 may be evaluated from

$$k_3 = i\, 2^{1/2} G \lim_{z \to z_1} (z - z_1)^{1/2} f'(z) \qquad (7)$$

In the mapped plane, Eq. (7) becomes

$$k_3 = i 2^{1/2} G \lim_{\zeta \to \zeta_1} [\omega(\zeta) - \omega(\zeta_1)]^{1/2} \frac{F'(\zeta)}{\omega'(\zeta)} \qquad (8)$$

where ζ_1 corresponds to z_1. Thus, a knowledge of $F'(\zeta)$ and $\omega(\zeta)$ in the vicinity of ζ_1 is sufficient to compute k_3.

As an example, consider the problem of an infinite body containing a circular hole of radius b with two collinear cracks of equal depth a originating at the edge of the hole. This configuration can be mapped onto the unit circle in the ζ plane by means of

$$\omega(\zeta) = Z + (Z^2 - b^2)^{1/2} \qquad (9)$$

where

$$Z = \frac{R}{2}\left(\zeta + \frac{1}{\zeta}\right) \qquad 2R = a + b + \frac{b^2}{a + b}$$

In this case, the crack tips $z = \pm(a + b)$ on the real axis transform to $\zeta = \pm 1$. At infinity, the body is subjected to longitudinal shear τ^∞, which makes an angle α with the x axis. By virtue of a simple analogy between this problem and the problem in plane hydrodynamics of potential flow,[5] it is possible to construct

$$F(\zeta) = \frac{\tau^\infty R}{G}\left[e^{-i\alpha}\zeta + \frac{e^{i\alpha}}{\zeta} \right] \qquad (10)$$

Inserting Eqs. (9) and (10) into Eq. (8) results in

$$k_3 = (a + b)^{-3/2}[(a + b)^4 - b^4]^{1/2}\tau^\infty \sin\alpha \qquad (11)$$

An approximate solution of the same problem in plane extension was given by Bowie.[6] As a special case of interest, when $b = 0$, Eq. (11) reduces to $\tau^{\infty}a^{1/2} \sin\alpha$, which is the solution of a single crack of length $2a$ under longitudinal shear. This limiting case also may be obtained upon substituting Eqs. (4) and (10) into Eq. (8) with $R = a/2$.

In closing, it should be pointed out that, in contrast to the complex variable method used in the plane theory of elasticity, the problem in longitudinal shear permits irrationality of the mapping function. Consequently, the present approach will allow the consideration of crack configurations heretofore avoided.

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Effect of Nodal Regression on Spin-Stabilized Communication Satellites

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A SPIN-STABILIZED satellite is useful for communication purposes if the antenna pattern is symmetric about the spin axis as in a so-called toroidal antenna where the unilluminated region is a cone about the spin axis with semivertex angle β, as shown in Fig. 1. The illuminated space consists of a triangle of revolution with an "antenna angle" 2α defined as shown in Fig. 1 by $\alpha = 90° - \beta$. The angle α will be referred to as the "semivertex angle" of the antenna. If the spin axis were to remain perpendicular† to the orbital plane at all times, the angle α needs never exceed the minimum value α_1, where (see Fig. 1)

$$\sin\alpha_1 = R_0/r \tag{1}$$

where R_0 is the earth radius and r is the geocentric satellite distance.

However, if the spin axis is initially perpendicular to the orbital plane, it will not remain so, since the orbit regresses while the spin-axis direction remains fixed in inertial space.‡ Thus, the actual antenna angle 2α must be increased by a certain tolerance angle $2\alpha_T$ in order for the antenna pattern
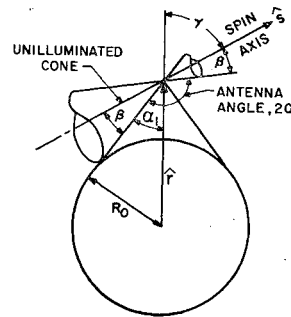
Fig. 1 Antenna shown casting no shadows.

to maintain full coverage of the visible earth surface. The authors now show how to find α_T for a given circular orbit.

Figure 2 shows, on a unit sphere, the original orbit of a satellite at inclination i with the equator. The pole of the original orbit is at S, and the unit vector $\hat{s} = \mathbf{OS}$ is parallel to the spin axis of the satellite which is chosen to be perpendicular to the original orbital plane. As time goes on, the node (intersection of orbital plane with earth equator) shifts from its original position N_1 through the angle Ω to a new position N_2. At the same time, the satellite may be anywhere in the perturbed orbit such as at position Q that has radius vector $\hat{r}$ from the center of the earth. The radius vector $\hat{r}$ lying in the orbital plane is normal to $\hat{p}$, the perpendicular to the perturbed orbit. The tip P of vector $\hat{p}$ moves through an angle Ω along the small circle at colatitude i.

In order to find the antenna half-angle α required for a given orbit, imagine a plane passed through the radius vector r and the spin vector $\hat{s}$, as shown in Fig. 1, from which it is evident that

$$\beta = \gamma - \alpha_1 \tag{2}$$

and

$$\alpha = 90° - \beta = 90° + \alpha_1 - \gamma \tag{3}$$

where γ is the acute angle between $\hat{r}$ and $\hat{s}$. The maximum value of α required anywhere in the orbit occurs where γ is minimum. From Fig. 2, it may be seen that the minimum value of γ occurs when $\hat{r}$ is in the plane of $\hat{p}$ and $\hat{s}$, at which point $\gamma = \gamma_{min}$ and

$$\gamma_{min} = 90° - \delta \tag{4}$$

or

$$\sin\gamma_{min} = \cos\delta = \hat{p} \cdot \hat{s} \tag{5}$$

In order to calculate δ, which is a sector of the great circle, through P and S, one may apply the cosine law of spherical trigonometry to the spherical triangle APS (A is the earth's pole) and find

$$\cos\delta = \cos^2 i + \sin^2 i \, \cos\Omega \tag{6}$$

The final expression for the required antenna half-angle is given by

$$\alpha = 90° + \alpha_1 - \text{arc } \sin(\cos^2 i + \sin^2 i \, \cos\Omega) \tag{7}$$

This expression is valid so long as α remains less than 90°. For values of i and Ω such that Eq. (7) predicts α greater than 90°, one must use $\alpha = 90°$ corresponding to an isotropic antenna. This may be verified from Fig. 1, which shows that when γ is less than α_1 the spin axis intercepts the earth, and the entire earth cannot be illuminated except by an isotropic antenna with $\alpha = 90°$. Thus, the condition for which a toroidal antenna can provide more intense illumination than an isotropic antenna with equal total power and complete earth coverage at all times is that δ is less than $90° - \alpha_1$, or

$$\cos\delta = \cos^2 i + \sin^2 i \, \cos\Omega > \sin\alpha_1 \tag{8}$$